Exercises 10, 29.04.2025 Solutions

10.1.

When the voltage is applied to the top and bottom surfaces of the bar, the current will flow along the axis of the beam. At the same time, the electric field generated at the application of the voltage, is not necessarily directed parallel to the axis of the beam. This can be shown writing the tensor of conductivity for the material considered in the problem. In the crystallographic reference frame, where Ox_3 axis is aligned with the symmetry axis, the $\bar{3}$ conductivity tensor has the form (see Symmetry Tables):

$$\tau = \begin{pmatrix} \tau_{11} & 0 & 0 \\ 0 & \tau_{11} & 0 \\ 0 & 0 & \tau_{33} \end{pmatrix}.$$

Thus, vector components of the electric field and of the electric current are connected by the following relation:

$$\begin{pmatrix} J_1 \\ J_2 \\ J_3 \end{pmatrix} = \begin{pmatrix} \tau_{11} & 0 & 0 \\ 0 & \tau_{11} & 0 \\ 0 & 0 & \tau_{33} \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} = \begin{pmatrix} \tau_{11}E_1 \\ \tau_{11}E_2 \\ \tau_{33}E_3 \end{pmatrix},$$

and a transverse voltage will appear.

In case where the vectors $\vec{J}$ and $\vec{E}$ are not parallel, i.e., where the electric field is not directed along the axis of the beam, a transverse voltage appears in the bar.

It is possible to show that there exist certain directions, cutting the bar at which will lead to parallel directions of the electric field and the electric current. At these directions, the transverse voltage will not appear in the system:

i) When the electric field is directed along the principal x_3 axis, i.e. $\vec{E} = (0,0,E_3)$:

$$\vec{J} = \begin{pmatrix} 0 \\ 0 \\ \tau_{33}E_3 \end{pmatrix} = \tau_{33} \cdot \begin{pmatrix} 0 \\ 0 \\ E_3 \end{pmatrix} = \tau_{33}\vec{E}$$

ii) When the electric field is directed along any direction perpendicular to the x_3 axis, i.e. $\vec{E} = (E_1, E_2, 0)$:

$$\vec{J} = \begin{pmatrix} \tau_{11}E_1 \\ \tau_{11}E_2 \\ 0 \end{pmatrix} = \tau_{11} \cdot \begin{pmatrix} E_1 \\ E_2 \\ 0 \end{pmatrix} = \tau_{11}\vec{E}$$

In all other cases, the electric current is not parallel to the electric field.

To conclude, the transverse voltage does not appear when the thin bar is cut in such a way that the longer dimension be either parallel or perpendicular to the symmetry axis

10.2

Experimentally, the heat capacity of the specimen is sought as

$$C_{(\mathrm{exp})} = \frac{\delta Q}{\delta T}$$

To find the relation between δQ and δT, we will use the constitutive equations written for absent electric field E_i:

$$\varepsilon_i = s_{ij}\sigma_j + \alpha_i \delta T,$$

$$\delta Q = T\alpha_i\sigma_i + C\delta T.$$

In case **(a)**, the sample is mechanically free, implying all $\sigma_i = 0$. Then, $\delta Q = C\delta T$, and

$$C_{(a)} = \frac{\delta Q}{\delta T} = C.$$

In case **(b)**, the sample is kept mechanically free in x_1 and x_2 directions, implying $\sigma_1 = \sigma_2 = \sigma_4 = \sigma_5 = \sigma_6 = 0$, and $\sigma_3 \neq 0$. The constitutive equation for δQ is simplified into:

$$\delta Q = T\alpha_3\sigma_3 + C\delta T.$$

To find σ_3, we use the constitutive equation for $\varepsilon_3 = 0$, which must not change during the measurement:

$$\varepsilon_3 = s_{33}\sigma_3 + \alpha_3\delta T = 0 \quad \Rightarrow \quad \sigma_3 = -\frac{\alpha_3}{s_{33}}\delta T,$$

$$\delta Q = T\alpha_3\sigma_3 + C\delta T = \left(C - T\frac{\alpha_3^2}{s_{33}}\right)\delta T,$$

$$C_{(b)} = \frac{\delta Q}{\delta T} = C - T\frac{\alpha_3^2}{s_{33}}.$$

Thus, in **(a)** and **(b)** the measured heat capacities are different. Specifically,

$$\frac{C_{(b)} - C_{(a)}}{C_{(a)}} = -\frac{T\alpha_3^2}{Cs_{33}} = -\frac{300 \cdot (3.5 \times 10^{-5})^2}{2.42 \times 10^6 \cdot 15.7 \times 10^{-12}} = -0.0097$$

The measured difference between heat capacities is less than 1% and, consequently, the impact of mechanical conditions on the heat capacity can be neglected.

10.3 $E = \alpha P + \beta P^3$

Extremes: $\frac{\partial E}{\partial P} = 0 = \alpha + 3\beta P^2$.

Extremes: $P_{ext}^2 = -\frac{\alpha}{3\beta}$

$P_S = \sqrt{-\frac{\alpha}{\beta}} \Rightarrow P_{ext} = \pm \frac{P_S}{\sqrt{3}}$

Now find E_c:

$$E_c = \alpha \frac{P_S}{\sqrt{3}} + \beta \frac{P_S^3}{3^{3/2}}$$

(remember $P_S^2 = -\frac{\alpha}{\beta}$)

$$E_c = \alpha\left(\frac{3P_S}{3\sqrt{3}} - \frac{P_S}{3\sqrt{3}}\right) = \frac{2\alpha P_S}{3\sqrt{3}}$$

$$\chi = \frac{\partial P}{\partial E} = \frac{1}{\alpha + 3\beta P^2}$$

at $E = 0 \Rightarrow \frac{\partial P}{\partial E} = -\frac{1}{2\alpha}$

$$\boxed{E_c = \frac{1}{3\sqrt{3}} \frac{P_S}{\chi}}$$